Journal of Engineering Physics and Thermophysics, Vol. 75, No. 5, 2002

CALCULATION OF A TWO-PHASE DYNAMIC LAMINAR BOUNDARY LAYER AND A THERMAL LAMINAR BOUNDARY LAYER ON A PLATE

A. P. Vasil'ev UDC 536.24:621.1

The approximate method of calculation of a bubble boundary layer based on the integral relations for momentum and energy is proposed. The corresponding equations are derived and results of the numerical investigation of heat exchange are given.

Consideration is given to the flow of a viscous conducting fluid with monodisperse ideal-gas bubbles in crossed homogeneous magnetic fields and electric fields. The effects of adhesion, fragmentation, and interaction between the bubbles, the energy of their random motion, and the capillary effects at the phase boundary are disregarded.

Of interest for engineering calculations are problems of flow past a solid surface and of heat exchange on it which can be solved within the framework of the notions of a boundary layer [1].

The approximate method of calculation of such flows is based on the employment of integral relations for momentum and energy [2] which have distinctions from their single-phase analogs as applied to two-phase flows. We derive these relations for laminar flow on a plane surface.

Having evaluated the order of terms in the equations of the momenta of the phases, for a plane steady-state flow on the plate $\mathbf{v}_i = \{u_i, v_i, 0\}$ we obtain

$$\frac{\partial}{\partial x}(\rho_1 u_1) + \frac{\partial}{\partial y}(\rho_1 v_1) = 0 \,, \tag{1}$$

$$\frac{\partial}{\partial x}(\rho_2 u_2) + \frac{\partial}{\partial y}(\rho_2 v_2) = 0 \,, \tag{2}$$

$$\rho_1 u_1 \frac{\partial u_1}{\partial x} + \rho_1 v_1 \frac{\partial u_1}{\partial x} = -\alpha_1 \frac{\partial p}{\partial x} + f_{1x} - F_{12x} + \frac{\partial \tau_1}{\partial y} \,, \tag{3}$$

$$\rho_2 u_2 \frac{\partial u_2}{\partial x} + \rho_2 v_2 \frac{\partial u_2}{\partial x} = -\alpha_2 \frac{\partial p}{\partial x} + F_{12x} \,. \tag{4}$$

In these equations, F_{12x} is the force of interaction between the phases caused by their velocity nonequilibrium and determined by the force-interaction model (electromagnetic expulsion, resistance to flow, the Magnus or Zhukowski force, the virtual-inertia force, etc.) [3].

In the case where the transverse forces are absent because of the velocity nonequilibrium the system of equations (1)–(4), by the methods of boundary-layer theory, is reduced to the form [4]

$$\frac{\partial}{\partial x}[\rho_1 u_1 (U_1 - u_1)] + \frac{\partial}{\partial y}[\rho_1 v_1 (U_1 - u_1)] + \rho_1 (U_1 - u_1)\frac{dU_1}{dx} = -\Delta F_{12} - \frac{\partial \tau_1}{\partial y} \,,$$

Orenburg State University, Orenburg, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 75, No. 5, pp. 74–80, September–October, 2002. Original article submitted October 22, 2001.

1062-0125/02/7505-1099$27.00 ©2002 Plenum Publishing Corporation

$$\frac{\partial}{\partial x}[\rho_2 u_2 (U_2 - u_2)] + \frac{\partial}{\partial y}[\rho_2 u_2 (U_2 - u_2)] + \rho_2 (U_2 - u_2) \frac{dU_2}{dx} = \Delta F_{12} .$$

Integrating these equations across the dynamic boundary layer between the limits from $y = 0$ to $y = \delta(x)$ with allowance for the rule of differentiation of the integral with respect to the parameter with a variable upper limit [5] and for the condition at the boundaries of the boundary layer

$$y = 0: \quad u_1 = 0 , \quad u_2 = 0 , \quad \tau_1 = \tau_{1w} , \quad v_1 = 0 , \quad v_2 = 0 ;$$

$$y = \delta(x): \quad u_1 = U_1 , \quad u_2 = U_2 , \quad v_1 = 0 , \quad v_2 = 0 , \quad \tau_1 = 0$$

we obtain

$$\frac{d}{dx}\int_0^{\delta(x)} \rho_1 u_1 (U_1 - u_1)\, dy + \frac{dU_1}{dx}\int_0^{\delta(x)} \rho_1 (U_1 - u_1)\, dy = \tau_{1w} - \int_0^{\delta(x)} \Delta F_{12}\, dy ,$$

$$\frac{d}{dx}\int_0^{\delta(x)} \rho_2 u_2 (U_2 - u_2)\, dy + \frac{dU_2}{dx}\int_0^{\delta(x)} \rho_2 (U_2 - u_2)\, dy = \int_0^{\delta(x)} \Delta F_{12}\, dy .$$

We introduce the following notation:

$$\delta_1^{**} = \int_0^{\delta(x)} \alpha_1 \frac{u_1}{U_1}\left(1 - \frac{u_1}{U_1}\right) dy , \quad \delta_2^{**} = \int_0^{\delta(x)} \alpha_2 \frac{u_2}{U_2}\left(1 - \frac{u_2}{U_2}\right) dy , \tag{5}$$

$$\delta_1^{*} = \int_0^{\delta(x)} \alpha_1 \left(1 - \frac{u_1}{U_1}\right) dy , \quad \delta_2^{*} = \int_0^{\delta(x)} \alpha_2 \left(1 - \frac{u_2}{U_2}\right) dy \tag{6}$$

i.e., the momentum thicknesses in the phases δ_i^{**} (5) and the displacement thicknesses δ_i^{*} (6) respectively. The integral relations for the momenta in the phases take the canonical form

$$\frac{d}{dx}\left(\rho_1^0 U_1^2 \delta_1^{**}\right) + \rho_1^0 U_1 \frac{dU_1}{dx} \delta_1^{*} = \tau_{1w} - \int_0^{\delta} \Delta F_{12}\, dy ,$$

$$\frac{d}{dx}\left(\rho_2^0 U_2^2 \delta_2^{**}\right) + \rho_2^0 U_2 \frac{dU_2}{dx} \delta_2^{*} = \int_0^{\delta} \Delta F_{12}\, dy .$$

We employ the ratio of the phase velocities in the potential part of the flow $U_2 / U_1 = S$, i.e., the slippage coefficient of the phases, and $\rho_1^0 / \rho_2^0 = \rho_*$, i.e., the reduced density. Then, adding together the previous equations, upon certain transformations we obtain the integral relation for the entire mixture

$$\frac{d\delta^{**}}{dx} + (2\delta^{**} + \delta^{*}) \frac{U_1'}{U_1} + \frac{\delta^{*}}{\rho_*} S S' = \frac{\tau_{1w}}{\rho_1^0 U_1^2} , \tag{7}$$

where the prime denotes the derivative with respect to x and the displacement thickness δ^{*} and the momentum thickness δ^{**} for the mixture are determined by the following expressions:

$$\delta^{**} = \delta_1^{**} + S^2 \frac{\delta_2^{**}}{\rho_*}, \quad \delta^{*} = \delta_1^{*} + S^2 \frac{\delta_2^{*}}{\rho_*}. \tag{8}$$

A distinctive feature of the integral relation (7) is that all the diversity of the force interaction between the phases in their velocity nonequilibrium is taken into account in it in terms of the characteristic S.

It should be noted that for S = const Eq. (7) coincides with the von Kármán integral equation in representation form [4].

We also derive the integral relation for the boundary-layer energy on the basis of the heat-inflow equation [3], which, in the boundary-layer approximation in the case of the steady-state flow of an incompressible fluid (subsonic velocities of flow) and with allowance for the temperature equilibrium in the phases, takes the form

$$\frac{\partial}{\partial x}[\rho_1 u_1 (T_\infty - T)] + \frac{\partial}{\partial y}[\rho_1 v_1 (T_\infty - T)] = \frac{1}{c_1^0}\frac{\partial q_{1y}}{\partial y} - \alpha_1 \frac{Q_\upsilon^0}{c_1^0} + \alpha_1 u_1 \frac{Q_{\upsilon,\infty}^0}{c_1^0 U_1},$$

where $Q_\upsilon^0 = Q_J^0 + Q_\mu^0 + Q_\upsilon^0$ is the density of the volume sources of heat release due to the Joule and viscous energy dissipations and to the velocity nonequilibrium of the phases.

For the bubble flow in which the volume content of the gas phase is low ($\alpha_2 \ll 1$ and $\rho_1^0 \gg \rho_2^0$), we can disregard the energy dissipation because of the velocity nonequilibrium of the phases. In this case, the density of the volume heat-release sources will be determined just by the Joule and viscous dissipations, $Q_\upsilon^0 = Q_J^0 + Q_\mu^0$. Outside the boundary layer, the viscous dissipation is absent and $Q_\mu^0 = 0$; then $Q_{\upsilon,\infty}^0 = Q_{J,\infty}^0$. Inside the boundary layer, we have $Q_\upsilon^0 = Q_J^0 + Q_\mu^0$; by virtue of the law of conservation of electric current, $Q_{J,\infty}^0 = Q_J^0$ = const and $Q_\mu^0 = \mu_1 (du_1/dy)^2$. Therefore, the last expression will be rewritten as

$$\frac{\partial}{\partial x}[\rho_1 u_1 (T_\infty - T)] + \frac{\partial}{\partial y}[\rho_1 v_1 (T_\infty - T)] = \frac{1}{c_1^0}\frac{\partial q_{1y}}{\partial y} - \alpha_1 \frac{Q_J^0}{c_1^0}\left(1 - \frac{u_1}{U_1}\right) - \frac{\alpha_1}{c_1^0}\mu_1 \left(\frac{du_1}{dy}\right)^2 .$$

Let us integrate this equation across the thermal boundary layer between the limits from $y = 0$ to $y = \delta_t(x)$, taking into account the conditions at its boundaries:

$$y = 0: \quad u_1 = v_1 = 0, \quad T = T_w, \quad q_{1y} = -\lambda_{ef}\frac{\partial T}{\partial y};$$

$$y = \delta_t: \quad u_1 = U_1, \quad v_1 = 0, \quad \frac{\partial T}{\partial y} = 0, \quad \frac{\partial u_1}{\partial y} = 0,$$

where λ_{ef} is the effective thermal conductivity of the two-phase flow, and pass to excess temperatures, having set $\Theta = T_\infty(x) - T(x)$ and $\theta = T(x) - T_w$. Then the integral relation will take the form

$$\frac{d\delta_t^{**}}{dx} + \left(\delta_t^{**} + \delta_t^{*}\right)\frac{Q_J^0}{\rho_1^0 c_1^0 U_1 \Theta} + \delta_t^{**}\frac{U_1'}{U_1} = \frac{\lambda_{ef}}{\rho_1^0 c_1^0 U_1 \Theta}\left.\frac{\partial \theta}{\partial y}\right|_{y=0} - \alpha_1 \frac{\mu_1}{\rho_1^0 c_1^0 U_1 \Theta}\int_0^{\delta_t}\left(\frac{du_1}{dy}\right)^2 dy, \tag{9}$$

where

$$\lambda_{ef} = \lambda_1^0 \left[1 + \frac{3}{2}\frac{\alpha_2}{1 - \sqrt[3]{\frac{9\pi}{16}\alpha_2^2}}\right]^{-1} \approx \lambda_1^0\left(1 - \frac{3}{2}\alpha_2\right).$$

and λ_1^0 is the true thermal conductivity of the carrier phase [6].

In the approximate method of calculation of the boundary layer, we prescribe the velocity and temperature profiles to integrate Eqs. (7) and (9). In the case of the dynamic boundary layer the profiles of phase velocities can be found with the following conditions at its boundary:

$$y = 0 : \quad u_1 = 0 \,, \quad u_2 = 0 \,, \quad \frac{\partial^2 u_1}{\partial y^2} = 0 \,, \quad \frac{\partial^2 u_2}{\partial y^2} = 0 \,;$$

$$y = \delta\,(x) : \quad u_1 = U_1 \,, \quad u_2 = U_2 \,, \quad \frac{\partial u_1}{\partial y} = 0 \,, \quad \frac{\partial u_2}{\partial y} = 0 \,,$$

which leads to the expressions [2]

$$\frac{u_1}{U_1} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 , \quad \frac{u_2}{U_2} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 . \tag{10}$$

We determine the temperature profile satisfying the conditions at the boundaries of the thermal boundary layer:

$$y = 0 : \quad \theta = 0 \,, \quad \frac{\partial^2 \theta}{\partial y^2} = 0 \,; \quad y = \delta_t\,(x) : \quad \theta = \Theta \,, \quad \frac{\partial \theta}{\partial y} = 0 \,,$$

which yields

$$\frac{\theta}{\Theta} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 . \tag{11}$$

For the prescribed profiles (10) and (11) and parameters U' and S' the system of Eqs. (7) and (9) can be integrated numerically. In the case of gradient-free flow past the plate this system allows an elementary solution.

Indeed, having set $U' = 0$ and $S' = 0$ in Eqs.(7) and (9), we rewrite them in the form

$$\frac{d\delta^{**}}{dx} = \frac{\mu_1}{\rho_1^0 U_1^2} \left.\frac{\partial u_1}{\partial y}\right|_{y=0} = \frac{3}{2}\frac{\mu_1}{\rho_1^0 U_1}\frac{1}{\delta\,(x)} \,, \tag{12}$$

$$\frac{d\delta_t^{**}}{dx} + \left(\delta_t^{**} + \delta_t^{*}\right)\frac{Q_J^0}{\rho_1^0 c_1^0 U_1 \Theta} + \alpha_1 \frac{\mu_1 U_1}{\rho_1^0 c_1^0 \Theta}\,\delta\,(x) = \frac{\lambda_{ef}}{\rho_1^0 c_1^0 U_1 \Theta} \left.\frac{\partial \theta}{\partial y}\right|_{y=0} . \tag{13}$$

In Eq. (13), it is assumed that $\delta_t < \delta$.

Let us consider first the solution of (12) for the dynamic boundary layer. The velocity profiles (10) enable us to compute the momentum thickness and with the condition $\rho_* > 1$ they yield ($\eta = y/\delta$)

$$\delta^{**} = \delta_1^{**} + \frac{S^2}{\rho_*}\delta_2^{**} \approx \delta_1^{**} = \alpha_1 \delta\,(x) \int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right) d\eta = \frac{39}{280}\alpha_1 \delta\,(x) \,.$$

Substituting this thickness into Eq. (12) and taking into account that $\delta = 0$ for $x = 0$, we obtain the solution

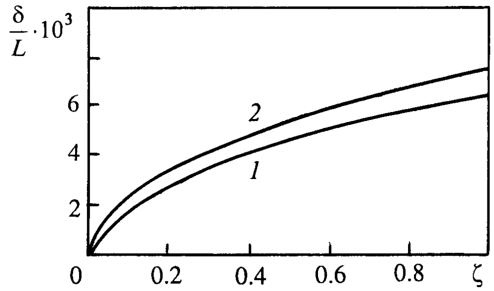


Fig. 1. Reduced thickness δ/L of the dynamic boundary layer vs. reduced length of the plate $\zeta = x/L$: 1) single-phase boundary layer; 2) two-phase boundary layer.

$$\delta(x) = \sqrt{\frac{1+\alpha_2}{1-\alpha_2}}\sqrt{\frac{280}{13}\frac{\nu_1^0 x}{U_1}}\ . \tag{14}$$

Here $\alpha_1 = 1 - \alpha_2 = \text{const}$.

When $\alpha_2 = 0$, the solution (14) becomes its single-phase analog [2]. It should be noted that (14) does not involve the forces responsible for the velocity nonequilibrium. The reason is that the information on interphase interaction is determined by the momentum thickness of the dispersed phase δ_2^{**}, which becomes insignificant because of the high value of the parameter ρ_*. Thus, the solution (14) can be called inertialess.

Figure 1 shows plots of the thickness of single-phase and two-phase dynamic boundary layers as a function of the reduced length of the plate. The thickness of the two-phase layer was calculated for a gas content of $\alpha_2 = 10\%$; the parameters of the liquid phase corresponded to those of water at a temperature of 50°C.

The given plots show that the dispersed phase in the carrier flow causes an increase in the thickness of the dynamic boundary layer.

In integrating the equation of the thermal boundary layer, we must take into account two possibilities: the thermal boundary layer is submerged in the dynamic one ($\delta_t < \delta$) and the thermal layer is thicker than the dynamic one ($\delta_t > \delta$).

Let us consider the first case where $\text{Pr} = \nu_1^0\rho_1^0 c_1^0/\lambda_1^0 \geq 1$, i.e., ordinary liquids.

If we introduce the criteria of the problem

$$\text{Po} = \frac{Q_J^0 L}{\lambda_1^0 \frac{\Theta}{L}}\ ,\quad \text{Pe} = \frac{U_1 L}{\frac{\lambda_1^0}{\rho_1^0 c_1^0}}\ ,\quad \text{Ec} = \frac{U_1^2}{c_1^0\Theta}\ ,\quad \text{Re} = \frac{\rho_1^0 U_1 L}{\mu_1^0}\ ,$$

Eq. (13) takes the form

$$\frac{d\delta_t^{**}}{dx} + \left(\delta_t^{**} + \delta_t^{*}\right)\frac{\text{Po}}{\text{Pe}}\frac{1}{L} + \alpha_1(1+\alpha_2)\frac{\text{Ec}}{\text{Re}}\frac{L}{\delta(x)}\int_0^{\delta_t/\delta}\left(\frac{d}{d\eta}\left(\frac{u_1}{U_1}\right)\right)^2 d\eta = \frac{3}{2}\frac{1-\frac{3}{2}\alpha_2}{\text{Pe}}\frac{L}{\delta_t(x)}\ . \tag{15}$$

Employing the velocity (10) and temperature profiles (11), we calculate the thicknesses and the integral involved in this expression.

Let $\delta_t/\delta = h < 1$ and $\eta = y/\delta$; then

$$\int_0^{\delta_t/\delta}\left(\frac{d}{d\eta}\left(\frac{u_1}{U_1}\right)\right)^2 d\eta = \int_0^{h}\left(\frac{d}{d\eta}\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\right)^2 d\eta = \frac{9}{4}h - \frac{3}{20}h^3 + \frac{9}{20}h^5\ ,$$

$$\delta_t^{**} = \int_0^{\delta_t} \alpha_1 \frac{u_1}{U_1}\left(1 - \frac{\theta}{\Theta}\right) dy = \frac{3}{20}\alpha_1 \delta h^2 \left(1 - \frac{1}{14} h^2\right), \quad \delta_t^{*} = \int_0^{\delta_t} \alpha_1 \left(1 - \frac{\theta}{\Theta}\right) dy = \alpha_1 \delta h \left(1 - \frac{3}{4} h + \frac{1}{8} h^2\right).$$

Substituting these expressions into the equation of the thermal boundary layer (15) and taking into account the solution (14), we obtain

$$\frac{dh}{d\zeta} = \frac{13}{280} \frac{1-\alpha_2}{1+\alpha_2} \frac{1}{\mathrm{Pr}} \frac{1 - \frac{3}{2}\alpha_2}{\zeta h(\zeta)\,\Psi(h)} - \frac{3}{40} \frac{1-\alpha_2}{\zeta \Psi(h)} - (1-\alpha_2) \frac{\frac{3}{20}\Phi(h) + \chi(h)}{\Psi(h)} \frac{\mathrm{Po}}{\mathrm{Pe}} - \frac{65}{1680}(1-\alpha_2)^2 \frac{\mathrm{Ec}}{\Psi(h)\,\zeta}, \qquad (16)$$

where ζ is the reduced longitudinal coordinate and the functions of h are defined by the following expressions:

$$\Phi(h) = h^2 \left(1 - \frac{h^2}{14}\right), \quad \Psi(h) = 2h\left(1 - \frac{h^2}{7}\right), \quad \chi(h) = h\left(1 - \frac{3}{4} h + \frac{h^2}{8}\right).$$

We find more exact conditions than $\mathrm{Pr} \geq 1$ [2] that ensure the condition of "submergence" of the thermal boundary layer in the dynamic one.

Indeed, if the right-hand side of Eq. (16) is negative at the point $\zeta = 0$, the function $h(\zeta)$ will become decreasing. Having redefined it for $x = 0$, i.e., having set $h(0) = 1$, we obtain the condition sought:

$$\mathrm{Pr} > \mathrm{Pr}_{cr1} = \frac{13}{21} \frac{1}{1+\alpha_2} \frac{1}{1 + 0.5\,\mathrm{Ec}}.$$

Let us consider integration of the equation of the thermal boundary layer (15) when its thickness is higher than the thickness of the dynamic layer, i.e., $\delta_t \geq \delta$. In this case we must take into account that $u_1 = U_1$ for $y > \delta$; then for the boundary-layer thicknesses we have

$$\delta_t^{**} = \int_0^{\delta_t} \alpha_1 \frac{u_1}{U_1}\left(1 - \frac{\theta}{\Theta}\right) dy = \alpha_1 \delta \left(-\frac{3}{8} + \frac{3}{8} h + \frac{3}{20}\frac{1}{h} + \frac{29}{1120}\frac{1}{h^3}\right),$$

$$\delta_t^{*} = \int_0^{\delta_t} \alpha_1 \left(1 - \frac{u_1}{U_1}\right) dy = \alpha_1 \int_0^{\delta} \left(1 - \frac{u_1}{U_1}\right) dy = \frac{3}{8}\alpha_1 \delta.$$

Taking into account that

$$\int_0^{\delta} \left(\frac{d}{dy}\left(\frac{u_1}{U_1}\right)\right)^2 dy = \frac{6}{5} \frac{1}{\delta(x)},$$

we reduce the equation of the thermal boundary layer to the form

$$\frac{dh}{d\zeta} = \frac{13}{280} \frac{1 - \frac{3}{2}\alpha_2}{1+\alpha_2} \frac{1}{\mathrm{Pr}} \frac{1}{\zeta h(\zeta)\,\Psi(h)} - \frac{\left(\frac{3}{8} + \Phi(h)\right)\left(\frac{1}{2} + \frac{\mathrm{Po}}{\mathrm{Pe}}\zeta\right)}{\Psi(h)\,\zeta} - \frac{39}{700}(1-\alpha_2) \frac{\mathrm{Ec}}{\Psi(h)\,\zeta}, \qquad (17)$$

where the functions of h are defined by the equalities

$$\Phi(h) = -\frac{3}{8} + \frac{3}{8} h + \frac{3}{20}\frac{1}{h} + \frac{29}{1120}\frac{1}{h^3}, \quad \Psi(h) = \frac{3}{8} - \frac{3}{20}\frac{1}{h^2} - \frac{87}{1120}\frac{1}{h^4}.$$

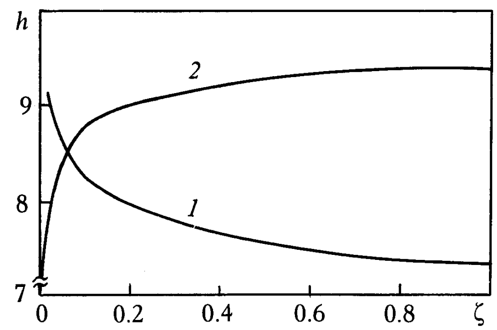
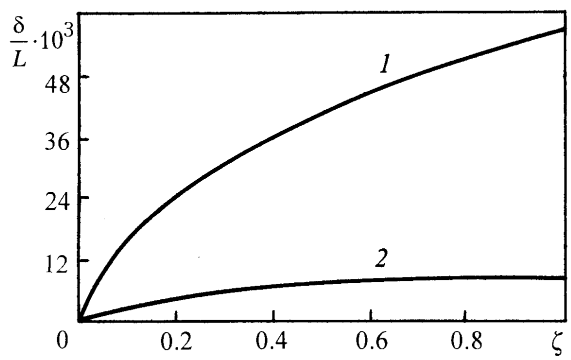


Fig. 2. Dependence of $h = \delta_t/\delta$ on the reduced length of the plate ζ in the two-phase boundary layer: 1) water (the scale of the ordinate axis is enlarged 10 times); 2) liquid gallium.



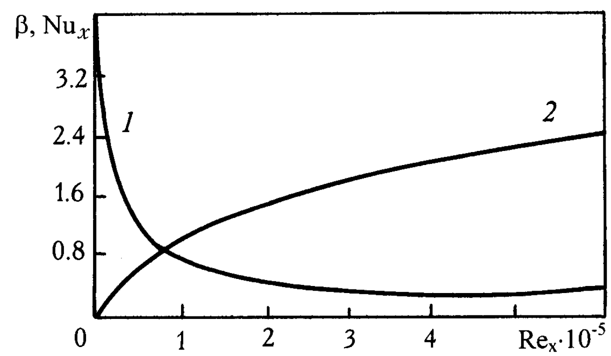


Fig. 3. Reduced thickness δ/L of the thermal (curve 1) and dynamic (curve 2) boundary layers vs. reduced length of the plate ζ (the dynamic layer is submerged in the thermal one).

Fig. 4. Local coefficient of heat transfer β/1000 (1) and Nusselt number $Nu_x/10$ (2) in the two-phase, liquid-metal flow vs. local Reynolds number.

By letting the derivative be positive at the point $\zeta = 0$ and redefining the function h in the manner mentioned above, we can find the conditions which must be satisfied by the criteria of the problem for the thermal layer to be thicker than the dynamic one ($\delta_t > \delta$):

$$\mathrm{Pr} < \mathrm{Pr}_{cr2} = \frac{26}{147}\,\frac{\dfrac{1 - \frac{3}{2}\alpha_2}{1 + \alpha_2}}{1 + 2\dfrac{\mathrm{Po}}{\mathrm{Pe}} + 0.2\,\mathrm{Ec}}\,.$$

Figure 2 gives the solutions of Eqs. (16) and (17) for two liquids: curve 1 corresponds to water at a temperature of 50^{o}C, while curve 2 corresponds to liquid gallium at a temperature of 100^{o}C with a volume content of air bubbles of α_2 = 10%.

The function $h(\zeta)$ was calculated for Po = Ec = 0 and Pr = 6.18 for water and for Pr = 0.02, $\Theta = 100^{o}$C, $\mathrm{Pe} = 1.12 \cdot 10^5$, $\mathrm{Ec} = 3.9 \cdot 10^{-7}$, and Po = 7.69 for gallium.

From the given plots it is clear that the condition of "submergence" of the thermal layer holds throughout the plate length in water, whereas the inverse condition holds in liquid gallium. The volume heat-release sources occurring in the flow of the liquid metal exert a slight influence on the quantity h for the indicated criteria of the problem. We can disregard the effects of the heat release Q_υ in their wide range. The critical Prandtl number was $\mathrm{Pr}_{cr1} = 0.563$ for curve 1 and $\mathrm{Pr}_{cr2} = 0.167$ for curve 2.

Figure 3 gives plots of the reduced thickness δ/L of the thermal (curve 1) and dynamic (curve 2) boundary layers for the two-phase, liquid-metal flow of gallium for a volume gas content of 10% and the parameters indicated above. The condition of "submergence" of the dynamic layer is ensured by small Prandtl numbers in liquid-metal flows [2].

Figure 4 gives plots of the local coefficient of heat transfer β (W/(m$^{2}\cdot{}^{o}C$)) and Nusselt number $Nu_x = \beta x/\lambda_1^0$ as functions of the local Reynolds number $Re_x = U_1 x/\nu_1^0$ which corresponds to the thermal boundary layer in Fig. 3. The character of change of the local heat-transfer coefficient is the same as for the analogous dependence in the single-phase liquid but the presence of the gas phase leads to a certain decrease in β.

It should be noted that the stability and convergence of the numerical algorithms of solution of the equation of the thermal boundary layer are disturbed on the interval of Prandtl numbers $Pr_{cr2} < Pr < Pr_{cr1}$.

The obtained equations of the thermal boundary layer can be employed in calculating the hydrodynamics and heat exchange in two-phase liquid-metal flows of certain power plants.

NOTATION

$\mathbf{v}_i$, vector of the velocity of the ith phase; x, y, Cartesian coordinates; ρ_i and ρ_i^0, reduced and true densities of the ith phase respectively; α_i, volume content of the ith phase; $f_{1x} = j_1 B$; reduced density of the electromagnetic force; j_1 and j_1^0, reduced and true densities of the electric current; $j_1 = \alpha_1 j_1^0$; B = const, magnetic-field induction; F_{12x}, longitudinal projection of the force of interaction between the phases because of velocity nonequilibrium; U_i and u_i, longitudinal velocities of the phases outside the boundary layer and inside it; v_i, transverse velocity of the ith phase within the boundary layer; τ_1 and τ_{1w}, viscous-friction stresses in the carrier phase and on the wall; δ^*, displacement thickness of the boundary layer; δ^{**}, momentum thickness for the entire mixture; δ and δ_t, thickness of the dynamic and thermal boundary layers; c_i^0, true specific mass heat of the ith phase of the flow at constant pressure; Q_υ^0, density of the volume heat-release sources; T and T_∞, temperatures of the flow inside the boundary layer and outside it; q_{1y}, heat-flux density, normal to the wall, in the liquid phase; λ, thermal conductivity; Θ and θ, excess temperatures; η, ζ, reduced coordinates; μ_1^0 and ν_1^0, true coefficients of dynamic and kinematic viscosities of the carrier phase; Po, Pe, Ec, Re, and Pr, Pomerantsev, Péclet, Eckert, Reynolds, and Prandtl numbers; L, characteristic dimension of the plate; h, dimensionless thickness of the thermal boundary layer; $\Phi(h)$, $\Psi(h)$, and $\chi(h)$, functions of the boundary layer. Subscripts and superscripts: i, phase index taking on the following values: 1, liquid (carrier) phase and 2, dispersed phase (gas bubbles); ∞, parameters of the flow outside the boundary layer; ef, effective value; w, parameters on the wall; cr, critical; J, Joule; t, thermal; 0, true value; x and y, projections onto the longitudinal and transverse axes of the Cartesian coordinates.

REFERENCES

1. L. G. Loitsyanskii, *Mechanics of Liquids and Gases* [in Russian], Moscow (1973).
2. I. L. Povkh, *Technical Hydromechanics* [in Russian], Leningrad (1976).
3. R. I. Nigmatulin, *Principles of the Mechanics of Heterogeneous Media* [in Russian], Moscow (1978).
4. H. Schlichting, *Boundary Layer Theory* [Russian translation], Moscow (1974).
5. G. M. Fichtenholz, *Differential- und Integralrechnung* [Russian translation], Vol. 2, Moscow (1969).
6. A. P. Vasil'ev, *Inzh.-Fiz. Zh.*, **39**, No. 4, 649–653 (1980).